The Matching Problem

a short journey through Algorithms and Polytopes

Maryam Salami
Former researcher at IASI-CNR, Marie Curie EU Program

Paolo Ventura
IASI-CNR

Outline of the webminars series

- The Matching Problem

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- The Matching Problem
- A combinatorial algorithm for bipartite graphs

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- The Edmonds algorithm for general graphs

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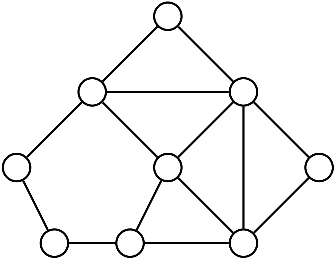
- The Edmonds algorithm for general graphs
- The matching polytope
- An interesting application on fullerene graphs

References

- A.M.H.Gerards *Matching*, Handbooks in Operations Research and Management Science, Volume 7, 1995, Pages 135-224.
- A. Schrijver *Combinatorial Optimization: Polyhedra and Efficiency*, (Vol. A), Springer, Berlin, 2003.

Matchings and Matching Problems

A *matching* of a graph $G = (V, E)$ is a set of pairwise disjoint edges of G. More formally, $M \subseteq E$ is a matching of G if and only if $\{s, t\} \cap \{u, v\} = \emptyset$ for each $\{s, t\}, \{u, v\} \in M$ with $\{s, t\} \neq \{u, v\}$.



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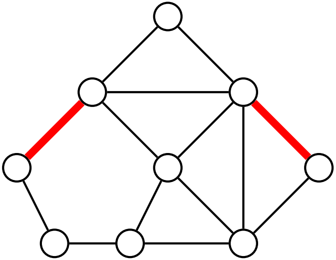

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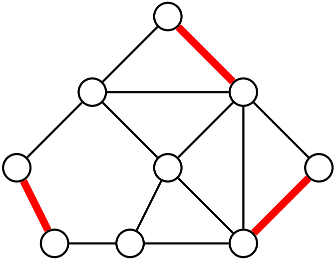

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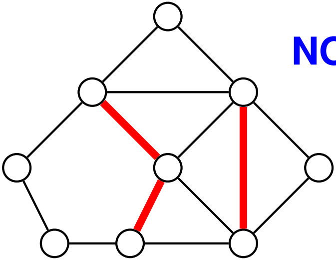

NO!

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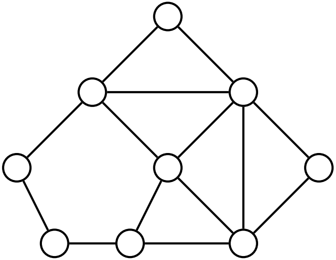

The empty set is a matching

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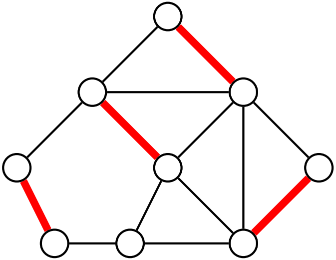

The Maximum Cardinality Matching Problem (MCMP) is to find a matching with the maximum number of elements

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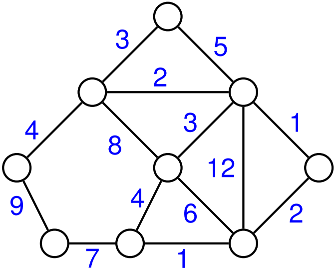

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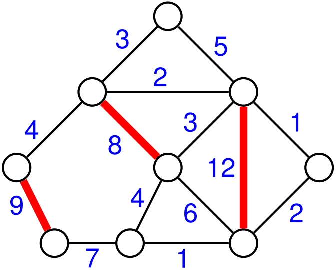
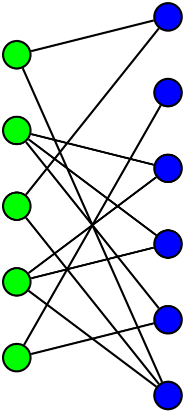


Given a cost function $c : E \to \mathbb{R}_+$

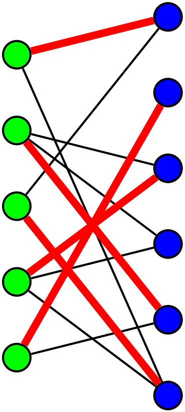
The Maximum Weight Matching Problem (MWMP) is to find a matching M^* with the maximum weight $c(M^*) = \sum_{e \in M^*} c(e)$

Applications



Assignment problems: green vertices must be assigned to blue vertices. Edges define compatibility.

Applications



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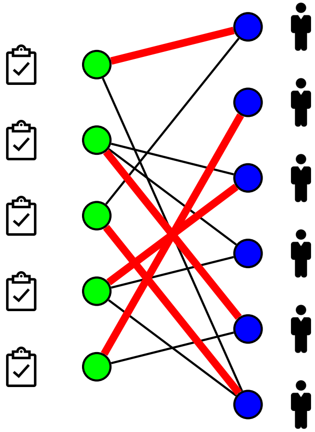

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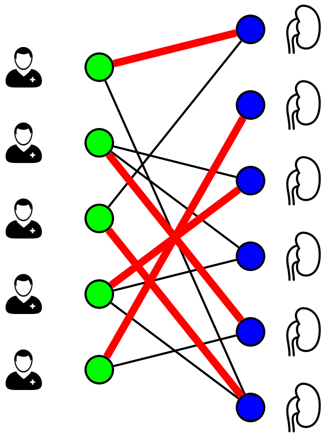

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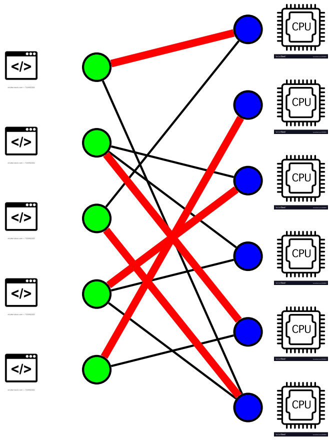

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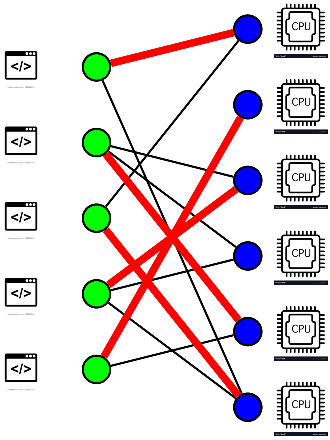

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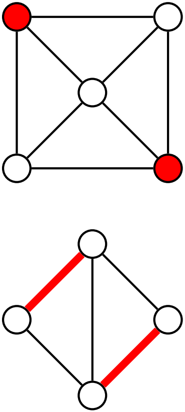
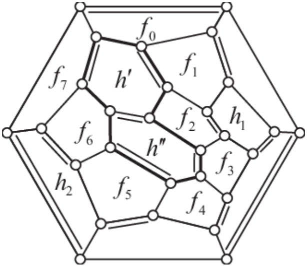
Maximizing the cardinality or an edge weight function.

Applications



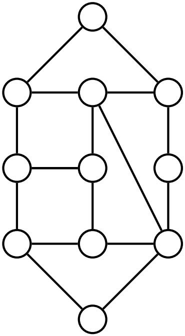
Combinatorics: the stable sets of a line graph G are the matchings of the root graph of G.

Applications



Chemistry: matchings can represent structural properties of molecules.

Bipartite graphs



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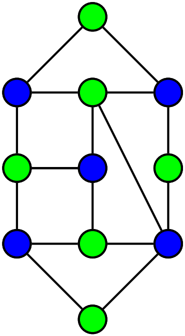

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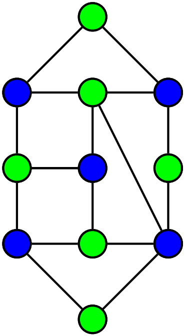

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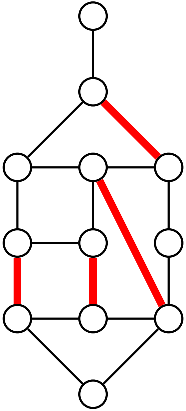
Theorem A graph $G = (V, E)$ is bipartite if and only if it does not contain odd cycles.

Augmenting paths



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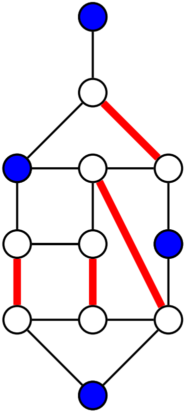

Given a graph $G = (V, E)$ with a matching M

A vertex $u \in V$ is *M*-exposed if it is not an extreme of any edge of M.

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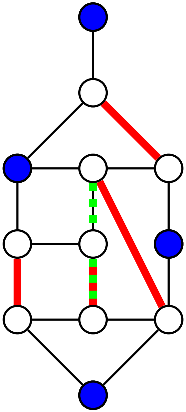

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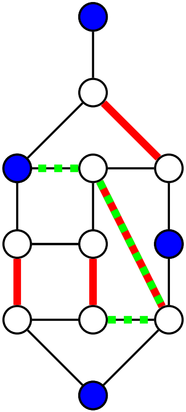

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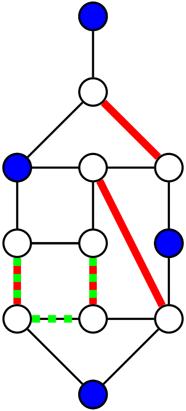

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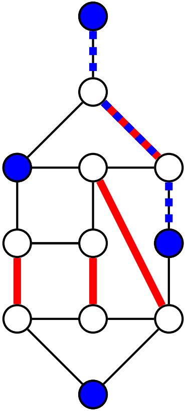

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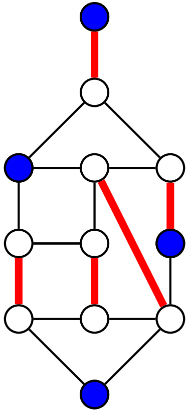

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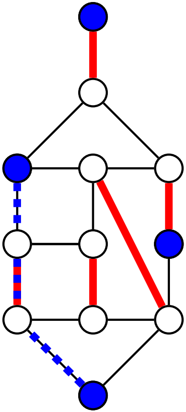

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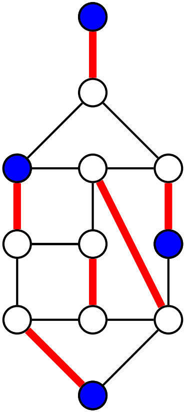

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- Let N be a matching with $|N| = |M^i| + 1$
- Let Q be a M^i-augmenting path in $N \triangle M^i$
- Then, $c(N) = c(N \triangle Q) + c(Q) \leq c(M^i) + c(P) = c(M^{i+1})$

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